UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE	:	**MATH7501**
ASSESSMENT PATTERN	:	**MATH7501A**
MODULE NAME	:	**Probability and Statistics**
DATE	:	**12-May-10**
TIME	:	**10:00**
TIME ALLOWED	:	**2 Hours 0 Minutes**

2009/10-MATH7501A-001-EXAM-145

TURN OVER

All questions may be attempted but only marks obtained on the best four *solutions will count.*

The use of an electronic calculator is *permitted in this examination.*

New Cambridge Statistical Tables are provided.

1. (a) Consider a sample space Ω with event space $\mathcal{F}$. State the three axioms of probability for a probability function $P(\cdot)$ defined on $\mathcal{F}$. Deduce from the axioms that, for arbitrary events A and B, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

   (b) Let A, B and C be mutually independent events. Explain what is meant by this statement. Deduce from this assumption that A and $B \cup C$ are independent events.

   (c) Let A and B be two events with $P(A) = 1/4$ and $P(B|A) = 1/3$. Deduce that $P(A \cup B) = 1/6 + P(B)$. If $P(A \cup B) = 1/3$, state whether A and B are independent.

2. Two factories manufacture identical components. On average, one component in 50 of those from factory A is defective, while one in 200 of those from factory B is defective. The components on sale in a particular shop have all come from the same factory, but it is not known which, and initially it is assumed that each is equally likely to be the supplier.

   (a) Suppose that a component from the shop is tested. What is the probability that it will be defective? If it is defective, what is the probability that the shop was supplied by factory A?

   (b) If the first component tested is defective, a second component is tested. What is the probability that the second one is also defective?

   If the second one is not defective, how must the probability that the shop is supplied by factory A be updated?

   (c) Suppose, as an alternative test strategy, that a batch of n components from the shop is tested and $k \, (0 \leqslant k \leqslant n)$ are found to be defective. Derive an expression in terms of n and k for the probability that the shop is supplied by factory A in this situation.

3. A discrete random variable X has a probability mass function given by

$$p_X(x) = \frac{kp^x}{x}, \quad \text{for } x = 1, 2, \ldots,$$

where p is a constant such that $0 < p < 1$ and k is a positive constant.

(a) Show that $k = -1/\ln q$ where $q = 1 - p$. Comment on the sign of k.

(b) Show that the probability generating function of X is given by

$$\Pi_X(t) = E(t^X) = \frac{\ln(1 - pt)}{\ln q}, \quad \text{for } |t| < 1/p.$$

Why must t be restricted to this range ?

(c) Show that

$$E(X) = -\frac{p}{q\ln q}$$

(d) Find $\mathrm{Var}(X)$.

4. (a) A continuous random variable X, taking non-negative values, has probability density function

$$f(x) = \frac{(\beta x)^\alpha}{x\Gamma(\alpha)} e^{-\beta x}, \quad x > 0,$$

for parameters $\alpha > 0$ and $\beta > 0$, where $\Gamma(\cdot)$ denotes the gamma function.

(i) Name this distribution, and state its mean and variance.

(ii) Show that the moment generating function (MGF) of this distribution is

$$M_X(t) = E(e^{tX}) = \left(1 - \frac{t}{\beta}\right)^{-\alpha}.$$

(b) Suppose that $X_1, \ldots, X_n$ are independent random variables such that X_i has the distribution given above with parameters α and β. Define $S = \sum_{i=1}^{n} X_i$.

(i) Write down an expression for the MGF of S. State the distribution of S and the parameters of the distribution.

(ii) If $n = 90$, $\alpha = 10$, and $\beta = 0.3$, use the Central Limit Theorem to approximate $P(S > 3100)$.

(You may use, without proof, the result that the MGF of a sum of independent random variables is the product of their individual MFGs.)

5. (a) Consider a statistic T which is an estimator of an unknown parameter θ. Denote the bias of the estimator T by $b(T,\theta)$ and the mean square error of T by $mse(T)$.

   (i) Define $b(T,\theta)$ and $mse(T)$.

   (ii) Show that
$$mse(T) = Var(T) + b^2(T,\theta).$$

   (b) Suppose Y has a binomial distribution with parameters n and p. Let
$$T = n\left(\frac{Y}{n}\right)\left(1-\frac{Y}{n}\right).$$

   (i) Write down the mean and variance of Y.

   (ii) Find $E(T)$. Hence, find an unbiased estimator for the variance of Y.

   (c) The number of breakdowns per week for a certain type of computer is a random variable Y having a Poisson distribution with mean λ

   (i) The weekly cost of repairing these breakdowns is $C = 5Y + Y^2$. Show that $E(C) = 6\lambda + \lambda^2$.

   (ii) Suppose $Y_1, Y_2, \ldots, Y_n$ is a random sample of observations of the weekly number of breakdowns. Show that $5Y_1 + Y_1^2$ and $\frac{5}{n}\sum_{i=1}^n Y_i + \frac{1}{n}\sum_{i=1}^n Y_i^2$ are unbiased estimators for $E(C)$. Which will you *guess* has the smaller variance?

6. In an experiment, 30 "standard" brand sparklers were burned and the burning time (in seconds) of each sparkler was measured. Fifteen of the sparklers were held vertically upwards while burning, and fifteen were held vertically downwards. For each group ($i = 1, 2$), the mean burning time ($\bar{x}_i$) and the sample variance (s_i^2) were as follows:

	Held upwards	Held downwards
$\bar{x}_i$	85	78
s_i^2	21	15

Let X_{ij} be the burning time of the jth sparkler in the ith group, $i = 1, 2$, $j = 1, \ldots, 15$. Assume that $X_{ij} \sim Normal(\mu_i, \sigma_i^2)$ independently for all i, j.

   (a) Describe the distributions of the sample variances S_1^2 and S_2^2 for the two groups. Describe the distribution of $S_1^2 + S_2^2$ if $\sigma_1^2 = \sigma_2^2 = \sigma^2$.

   (b) Test, at the 5% level, the hypothesis that the variances of the two groups' burning times are the same assuming that the true means are unknown.

   (c) Test, at the 5% level, the hypothesis that the mean burning times of the two groups are the same under the assumption that the variances are the same. Calculate a 95% confidence interval for $\mu_1 - \mu_2$.